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T. IMPROPER SOLUTIONS UNDER EXISTENCE ASSUMPTIONS: AN EXAMPLE

by

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IMPROPER SOLUTIONS UNDER EXISTENCE ASSUMPTIONS: AN EXAMPLE*

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J. H. Eaton**

ABSTRACT

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A pursuit problem is considered in which a pursuer X and target Y are points in an n-dimensional Euclidean space E^n. The pursuer is assumed to have knowledge of the control (and hence trajectory) to be used by the target and is to choose a control to minimize the time required to intercept the target. We will call this minimum time the pursuit time. The target, wishing to delay interception as long as possible desires to choose a control which maximizes the pursuit time. It is shown that if the target assumes the existence of an optimal control (i.e., a control for which the least upper bound of the pursuit time is attained) he can be lead to a choice of control which will result in an interception in less time than if he applied no control at all.

Author

FORMULATION

We will consider the pursuit problem considered by D. L. Kelendzheridze[1] in the form presented by Pontryagin et al, in reference 2. The notation and formulation used by Pontryagin et al is followed as closely as possible. In this problem the pursuer X and target Y are points in an n-dimensional space E^n. Let us denote the control parameter, control region, and trajectory of the pursuer by u, U, and x(t) respectively. Similar quantities for the target point will be denoted by v, V, and y(t). Let x(t) and y(t) be trajectories corresponding to initial conditions $x(0) = x_o$, $y(0) = y_o$ and to a pair of admissible controls u(t) and v(t). Any time t_i for which $x(t_i) = y(t_i)$ will be called an encounter time. If, for a particular x_o, y_o, v(t), and u(t) an encounter occurs, u(t) will be called a pursuing control and the smallest $t > 0$ for which an encounter occurs will be called the pursuit time. Fixing the initial positions x_o and y_o let us denote the pursuit time corresponding to v(t) and a pursuing control (u(t) by T_{uv}. From this point on we will assume that there exists at least one pursuing control u(t) for each admissible target control v(t). This

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restriction is necessary for if there is a target control v(t) for which there is no pursuing control the problem reduces to a somewhat different one of finding such a target control. We will also assume (Assumption I) that for any given admissible target control v(t) there exists an admissible pursuing control u(t) such that the corresponding pursuit time T_{uv} takes on its minimal value, which we denote by T_v. This assumption is not particularly restrictive and is satisfied for example if the target trajectories corresponding to admissible target controls are differentiable with respect to t and if the set of points on E^n reachable by the pursuer from x_0 in a length of time not exceeding t is closed and convex and if its boundary moves continuously with t. At this point Pontryagin et al assume (Assumption II) there exists an admissible target control v(t) which brings about the least upper bound of the values T_v, and they denote this least upper bound by T. Thus Assumption II:

$$T = \max_v T_v = \max_v (\min_u T_{u,v})$$

Assumption II will prove to be very restrictive. Proceeding under Assumptions I and II the problem becomes one of finding a pair of admissible controls u(t) and v(t) such that $T_{uv} = T$. Such a pair of controls is called an optimal pair of controls, and the corresponding pair of trajectories x(t) and y(t) is called an optimal pair of trajectories. In words, the problem is roughly as follows. The target knows that for each control v(t) he chooses, the pursuer, with a knowledge of v(t), will choose an admissible control u(t) to intercept the target in the minimum time T_v. Thus the target, wishing to avoid interception as long as possible, would like to find an admissible control v(t) which results in the pursuit time T_v achieving its least upper bound T. We will show later that disasterous results can be obtained if the target assumes that such a control exists.

Proceeding with the formulation as given by Pontryagin et al let the motion of the pursuer in the n-dimensional space E^n be described by the linear differential equation

$$dx/dt = f(x,u) = Ax + Bu + c \tag{1}$$

where A is an nxn matrix of constants, u, the control vector, belongs to the r-dimensional control region U which is a closed convex bounded polyhedron in E^r, and c, an n-vector, is a known function of time. Similarly, the motion of the target point is assumed to be described by the vector equation

$$dy/dt = g(y, v, t) \tag{2}$$

where y is an n-vector, g is a vector function which, in each of its componets, is continuous in y, v, and t and continuously differentiable with respect to the coordinates $y_1, y_2, \ldots, y_n$ of y. The control region V is taken to be a set in the s-dimensional space of the variable $v = (v^1, v^2, \ldots, v^s)$. The class of admissible controls for u and v is taken to be the set of piecewise continuous controls in U and V respectively. To solve the given problem, Pontryagin et al introduce two auxiliary vectors $\psi = (\psi_1, \ldots, \psi_n)$ and $\chi = (\chi_1, \ldots, \chi_n)$ as well as the two Hamiltonian functions

$$H_1(\psi, x, u) = \sum_{\alpha=1}^{n} \psi_\alpha f^\alpha(x, u) = \langle \psi, f(x, u) \rangle$$

$$H_2(\chi, y, v) = \sum_{\alpha=1}^{n} \chi_\alpha g^\alpha(y, v, t) = \langle \chi, g(y, v, t) \rangle$$

Using H_1 and H_2 two systems of equations for the auxiliary unknowns ψ_i and χ_i are written

$$d\psi_i/dt = -\partial H_1/\partial x^i \qquad i = 1, 2, \ldots, n \tag{3}$$

$$d\chi_i/dt = -\partial H_2/\partial y^i \qquad i = 1, 2, \ldots, n \tag{4}$$

When u(t), x(t), v(t), and y(t) are given and substituted into the right hand sides of (3) and (4) the resulting system of equations (3) and (4) are linear in ψ and χ. Any solution $\psi(t)$, $\chi(t)$ of these linear equations is said to be a solution corresponding to u(t), x(t), v(t), and y(t). Theorem 21*, proven in reference 2, gives a necessary condition of optimality for the problem under consideration, i.e., with Assumptions I and II.

Theorem 21 Let u(t) and v(t) be an optimal pair of controls, let x(t) and y(t) be the corresponding optimal pair of trajectories (see Eqs. (1) and (2)), and let T be the pursuit time. Then, there exist nontrivial solutions $\psi(t)$

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*This theorem is due to D. L. Kelendzheridze.

and $\chi(t)$ of systems (3) and (4) which correspond to u(t), x(t), and y(t) such that:

1^o the maximum conditions

$$\max_{u \epsilon U} H_1 (\psi(t), x(t), u) = H_1(\psi(t), x(t), u(t)), \tag{5}$$

$$\max_{v \epsilon V} H_2(\chi(t), y(t), v) = H_2(\chi(t), y(t), v(t)) \tag{6}$$

hold for all t, $0 \leq t \leq T$:

2^o at the time $t = T$, the conditions

$$H_1(\psi(T), x(T), u(T)) \geq H_2(\chi(T), y(T), v(T)), \tag{7}$$

$$\psi(T) = \chi(T) \tag{8}$$

hold.

ANALYSIS

We now apply Theorem 21 to a specific example. Let the motion of the pursuer be described by the equation

$$\frac{dx}{dt} = Ax + u, \qquad x(0) = 0 \tag{9}$$

where x is an n-vector, the control vector u is an n-vector and $A = -A'$ is an nxn matrix of constants. Here, a prime denotes transpose. The control region U is the circular region $||u||^2 = (u^1)^2 + (u^2)^2 \leq 4$. Note that U is not a polyhedron as was assumed in the derivation of Theorem 21, however, the crucial property of U that was used in Theorem 21 was that the set $\sum_{to}$ of states reachable by the pursuer from x_o in a time not exceeding t_o using an admissible control (i.e., a piecewise continuous control u(t) for which $u(t) \mathcal{E} U$ $0 \leq t \leq t_o$) was closed and convex. This property holds for the control region U. Let the motion of the target by dexcribed by the equation

$$\frac{dy}{dt} = B(y-a) + v \qquad y(0) = a \tag{10}$$

where $B = -B'$ is an nxn matrix of constants y is an n-vector a, an n-vector is constant and the control region V for the control vector v is the circular region $||v|| \leq 1$. Each of the systems (9) and (10) is a special case of the norm-invariant systems considered by Athans, Falb and Lacoss[3] and is chosen so that an analytic solution to the problem under consideration may be obtained. We will consider the problem described above for two values of the vector a. In one case, an optimal control pair exists, in the other an optimal control pair does not exist. For the problem under consideration we find

$$H_1(\psi, x, u) = \sum_{\alpha=1}^{n} \psi_\alpha f^\alpha(x,u) = \langle \psi, (Ax+u) \rangle$$

and (11)

$$H_2(\chi, y. v) = \sum_{\alpha=1}^{n} \chi_\alpha g^\alpha(y,v) = \langle \chi, B(y-a)+v \rangle$$

Whence, Eqs. (3) and (4) become, in vector form

$$\frac{d\psi}{dt} = -A'\psi$$

$$\frac{d\chi}{dt} = -B'\chi$$

from which we obtain

$$\psi(t) = e^{-A't}\psi(0)$$
$$\chi(t) = e^{-B't}\chi(0) \tag{12}$$

Let us now <u>assume</u> that an optimal control pair exists and apply Theorem 21. Then, if the pursuit time is T we find, combining (8) and (12) that a necessary condition for a control pair to be optimal is

$$\psi(t) = e^{-A'(t-T)}\psi(T)$$

and (13)

$$\chi(t) = e^{-B'(t-T)}\psi(T)$$

Here we have used the fact that $\psi(0) = e^{A'T}\psi(T)$ and $\chi(0) = e^{B'T}\chi(T)$. For convenience, let us choose $\psi(T)$ to be a unit vector. Using (13) in (5) and (6) we obtain the necessary conditions

$$\langle e^{-A'(t-T)}\psi(T),\ Ax(t) + u(t) \rangle = \max_{\|u\| \le 2} \langle e^{-A'(t-T)}\psi(T), Ax(t) + u \rangle$$

and (14)

$$\langle e^{-B'(t-T)}\psi(T), B[y(t)-a] + v(t) \rangle = \max_{\|v\| \le 1} \langle e^{-B'(t-T)}\psi(T), B[y(t)-a] + u \rangle$$

which imply

$$u(t) = 2e^{-A'(t-T)} \psi(T)$$

and (15)

$$v(t) = e^{-B'(t-T)} \psi(T)$$

Note that since A = -A' and B = -B',

$$\|e^{-A'(t-T)} \psi(T)\| = \|e^{-B'(t-T)} \psi(t)\| = \|\psi(T)\|$$

Using the controls given in (15) in systems (9) and (10) we obtain

$$\begin{aligned} x(t) &= 2te^{A(t-T)} \psi(T) \\ y(t) &= a + te^{B(t-T)} \psi(T) \end{aligned} \tag{16}$$

and hence

$$\begin{aligned} x(T) &= 2T \psi(T) \\ y(T) &= a + T\psi(T) \end{aligned} \tag{17}$$

Recalling that at the time of interception $x(T) = y(T)$ we find from (17)

$$2T\psi(T) = a + T\psi(T),$$

a condition which, since $\|\psi(T)\| = 1$, can be satisfied if and only if

$$\psi(T) = a/\|a\| = \bar{a}$$

and (18)

$$T = \|a\|$$

Note that (7) is satisfied for this choice of T and $\psi(T)$. Since the controls determined by (18) are the only controls which satisfy the necessary conditions of Theorem 21, they must be optimal <u>if</u> an optimal control pair exists. Thus, if an optimal control pair exists it is given by

$$\begin{aligned} u(t) &= 2e^{-A'(t-\|a\|)} \bar{a} \\ v(t) &= e^{-B'(t-\|a\|)} \bar{a} \end{aligned} \tag{19}$$

The corresponding pair of "optimal" trajectories is

$$\begin{aligned} x(t) &= 2te^{A(t-\|a\|)} \bar{a} \\ y(t) &= a + te^{B(t-\|a\|)} \bar{a} \end{aligned} \tag{20}$$

In order to simplify what follows let us restrict ourselves to a two-dimensional system and set

$$A = B = \begin{pmatrix} 0 & -\pi \\ +\pi & 0 \end{pmatrix}$$

Recall we have previously chosen U to be the circular set determined by $\|u\|^2 \leq 4$ and V to be the circular region $\|v\| \leq 1$.

Consider first case I, in which $a = \text{col}\,(1, 0)$. Then since

$$e^{At} = \begin{pmatrix} \cos \pi t & -\sin \pi t \\ +\sin \pi t & \cos \pi t \end{pmatrix}$$

we have directly from (20) the "optimal" trajectories

$$x^1(t) = 2t \cos \pi(t-1),\; x^2(t) = +2t \sin \pi(t-T),\; 0 \leq t \leq 1$$

and

$$y^1(t) = 1 + t \cos \pi(t-1),\; y^2(t) = +t \sin \pi(t-1)$$

or in polar form, r_x, θ_x, the trajectory of the pursuer is given by

$$r_x(t) = 2t,\; \theta_x(t) = \pi(1 + t). \tag{21}$$

For the target let us set $z(t) = y(t) - a$, then $z(t)$ is described in polar form by

$$r_z(t) = t,\; \theta_z(t) = \pi(1 + t) \tag{22}$$

The trajectories $x(t)$ and $y(t)$ are shown in Figure I, as well as $x(t_i)$, $y(t_i)$ and the sets of reachable points $\sum_{t_i}$ and $S_{t_i, a}$, for various values of $t_i \leq 2$.
We now seek to determine if these trajectories are optimal. To answer this question let us consider the set $\sum_t$ of points in E^n reachable from the origin by the pursuer in a time not exceeding t, and the set $S_{t,a}$ of points in E^n reachable from a by the target in a time not exceeding t.
It is easily verified that $\sum_t$ is the set of points whose Euclidean distance from the origin is equal to or less than 2t, and that $S_{t,a}$ is the set of points whose distance from a does not exceed t. Thus

$$\sum_t = \left\{ x \,\middle|\, \|x\| \leq 2t \right\}$$

$$S_{t,a} = \left\{ y \,\middle|\, \|y - a\| \leq t \right\}.$$

We observe that for $a = \text{col}(1, 0)$, $S_{1,a} \subset \sum_1$ (see Figure I), and hence there is no target control which will yield a pursuit time greater than 1. We must now check to see that the pursuit time $T = 1$ is actually attained with the choice of control v(t). Thus we must check to see that $y(t_i)$, does not belong to $\sum_{t_i}$ for any t_i less than 1. It is clear from Figure I (and it can be verified analytically) that $t = 1$ is the smallest value of t for which y(t) belongs to $\sum_t$. Consequently the pursuer cannot intercept the target on the trajectory y(t) in a time less than 1 and the control pair obtained using Theorem 21 is in fact optimal. Let us next consider Case II in which $a = \text{col}(2, 0)$ instead of col (1, 0) as in Case I. Eqs. (18) and (20) now yield $T = 2$ for the pursuit time and

$$\begin{aligned} x^1(t) &= 2t\cos\pi(t-2), \; x^2(t) = 2t\sin\pi(t-2) \\ y^1(t) &= 2 + t\cos\pi(t-2), \; y^2(t) = t\sin(t-2) \end{aligned} \quad 0 \le t \le 2 \tag{23}$$

for the "optimal" pair of trajectories. Trajectories (23) become, in polar form

$$\begin{aligned} r_x(t) &= 2t, \; \theta_x(t) = \pi t \\ r_z(t) &= t, \; \theta_z(t) = \pi t \end{aligned} \tag{24}$$

where $z(t) = y(t) - a$. The trajectories (23) are shown as the solid trajectories in Figure II. $x(t_i)$, $y(t_i)$, $\sum_{t_i}$ and $S_{t_i,a}$ are shown for various values of the parameter t_i. Let us now determine if the trajectories x(t) and y(t) as given in (23) are optimal. From Figure II it is clear that $S_{2,a} \subset \sum_2$ and hence that there is no admissible target control which results in a pursuit time greater than 2. However, in this case it is immediately obvious from Figure II that if the target follows the trajectory y(t) given in (23) the pursuit time $T = 2$ will not be attained since there are values to t_i smaller than 2 for which $y(t_i)$ belongs to $\sum_{t_i}$, e.g. $t_i = 1$. The pursuer can intercept the target the first time t_i for which $y(t_i)$ belongs to $\sum_{t_i}$, which is the first $t_i > 0$ for which $d(y(t_i), \sum_{t_i})$, the distance from $y(t_i)$ to $\sum_{t_i}$, is zero. The sets $\sum_{t_i}$ are are circular and centered about the origin, hence

$$\begin{aligned} d(y(t_i), \textstyle\sum_{t_i}) &= \|y(t_i)\| - 2t_i \\ &= \left[4 + 4t_i\cos\theta_z(t_i) + t_i^2\right]^{1/2} - 2t_i \end{aligned} \tag{25}$$

Here we have used the subscript i on t to indicate that we are speaking of points on the trajectory y(t) rather than the trajectory itself. Since we are interested in values of t_i in the interval $0 \leq t_i \leq 2$ for which $d(y(t_i), \sum_{t_i}) = 0$ we are essentially interested in the real roots of

$$3t^2 - 4t\cos \pi t - 4 = 0 \tag{26}$$

in the interval $0 \leq t \leq 2$. Equation (26) has three real roots in this interval, $t_1 = 0.76$, $t_2 = 1.76$, and $t_3 = 2$. The pursuer can intercept the target on the trajectory y(t) at any time in the interval $t_1 \leq t \leq t_2$, or at time $t_3 = 2$, consequently the pursuit time is t_1. Note that if the target were to apply zero control, and hence remain at $y = \mathrm{col}(2, 0)$, the resulting pursuit time would be $T = 1$. The target's assumption that an optimal control exists leads in this case to a control that is far worse than doing nothing. In fact we could have chosen a model for which the existence assumption would lead the target to a choice of control resulting in his interception in the least possible time of all choices. Consider now the admissible target trajectories $y_\phi(t) = a + z_\phi(t)$ where $z_\phi(t)$ is determined in polar form by

$$r_z(t) = t, \ \theta_z(t) = \pi t + \phi \tag{27}$$

We can investigate the times t_i for which $y(t_i)$ belongs to $\sum_{t_i}$ by substituting $\theta_z(t) = \pi t + \phi$ for πt in the cosine term of Eq. (26). The real roots of (26) in the interval $0 \leq t \leq 2$ are plotted as a function of u in Figure III. The heavy line in Figure III corresponds to the pursuit time for the trajectory $y_\phi(t)$. This pursuit time will be denoted by T_ϕ. Observe that the trajectory $y_{\pi/2}(t)$ results in a pursuit time $T_{(\pi/2)} = t_3(\pi/2) \approx 1.74$. In the interval shown in Figure III $t_3(\phi)$ is a monotonic decreasing function of ϕ. Thus as long as $T_\phi = t_3(\phi)$, an improved control is obtained by reducing ϕ, however, for $\phi \leq \phi_o$, $T_\phi = t_1(\phi) << t_3(\phi)$. Thus there does not exist an optimal trajectory of the form $y_\phi(t)$ since for $\epsilon > 0$, $T_{\phi_o} + 2\epsilon < T_{\phi_o} + \epsilon$ but $T_{\phi_o} = t_1(\phi_o) << T_{\phi_o} + \epsilon$. The trajectory $y_{\phi o}(t)$ is represented by the dashed line in Figure II. It can be shown that there is no admissible control which results in a pursuit time $T_{\phi o}$ or less, and hence that no optimal control exists for the problem under consideration.

CONCLUSIONS

The target's problem is essentially as follows: Find an admissible control v(t) for which the corresponding trajectory y(t) avoids the closed set Σ_t as long as possible. When the problem is phrased in this manner it is clear that one should not in general expect an optimal solution to exist since the problem is now one involving trajectory constraints for which the region of allowable trajectories is not closed (since it is the complement of the closed set Σ_t). The assumption that an optimal control pair exists is equivalent to assuming, among other things, that the pursuit time T is the smallest t for which $S_t \subset \Sigma_t$ and that it is possible to find a trajectory y(t) corresponding to an admissible control v(t) for which $y(T) \in \partial S_T \cap \partial \Sigma_T$, and such that $y(t) \cap S_t$ is empty, $0 \leq t \leq T$. Here ∂S_T denotes the boundary of S_T and $\partial \Sigma_T$ denotes the boundary of Σ_T. In conclusion, we note that the assumption of the existence of an optimal control pair is very restrictive, and that there seems in general to be no a priori way of determining if an optimal control pair exists.

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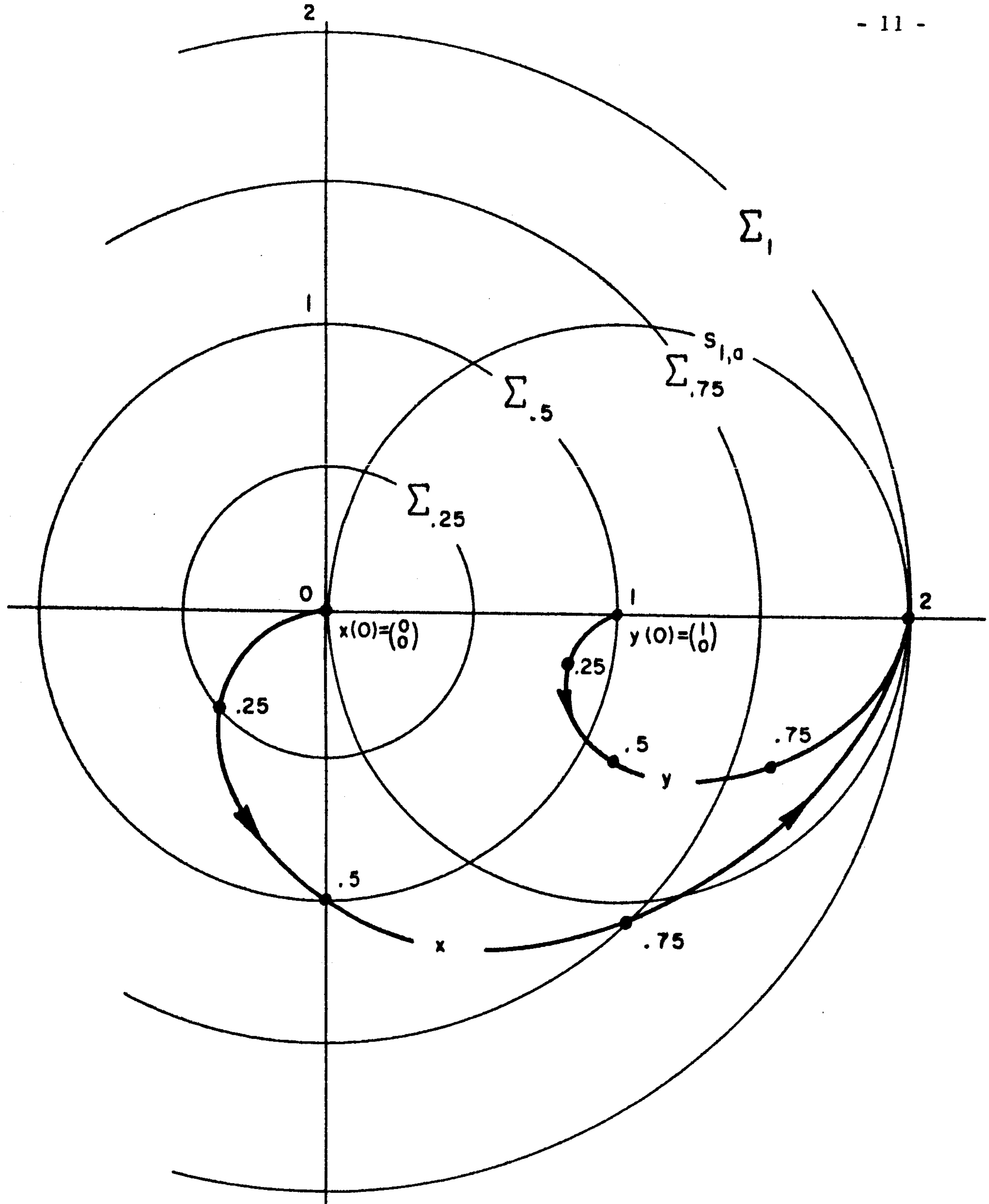

Fig. I "Optimal" trajectories for $a=\binom{1}{0}$

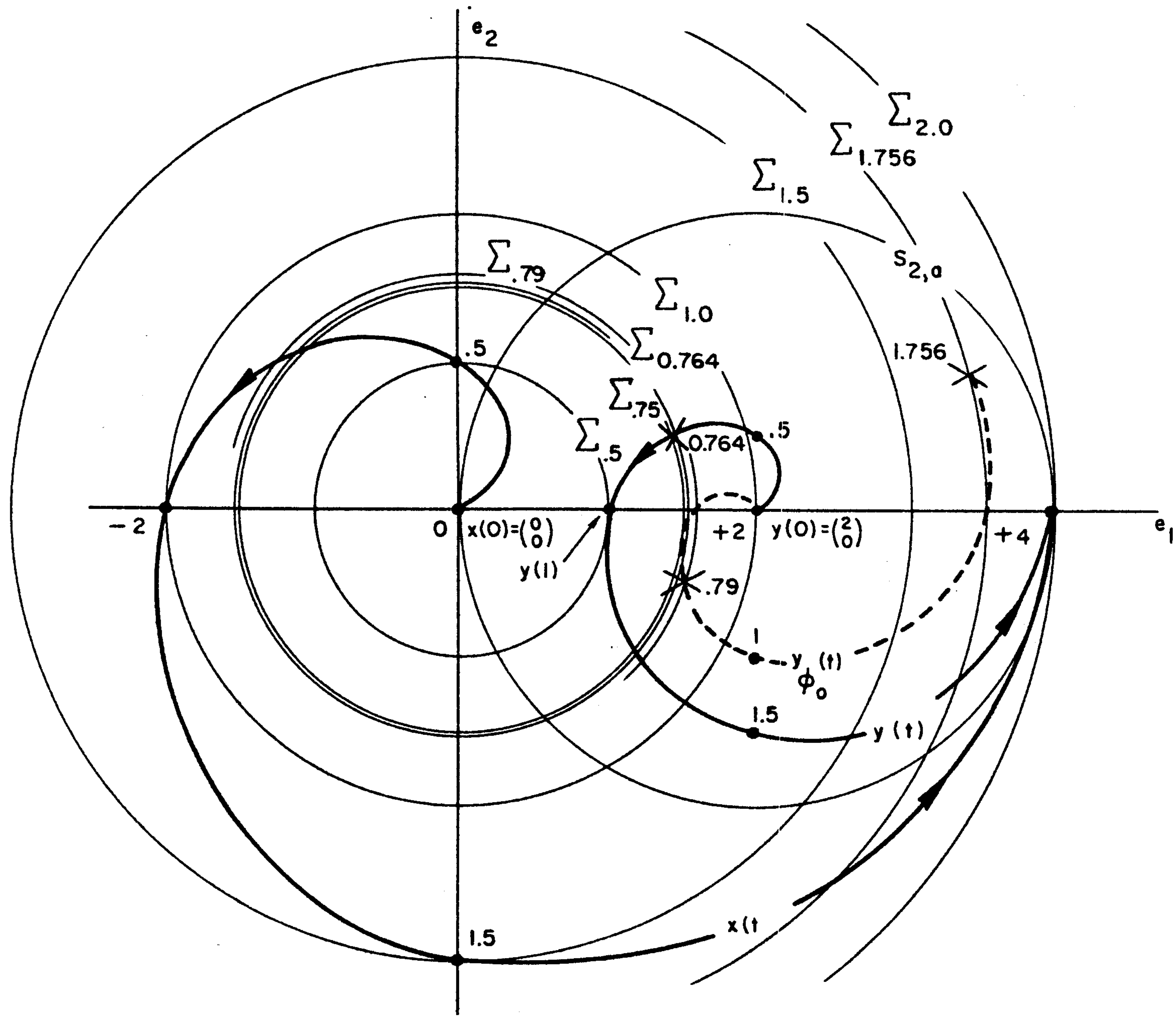

Fig.II "Optimal" trajectories for $a=\binom{2}{0}$

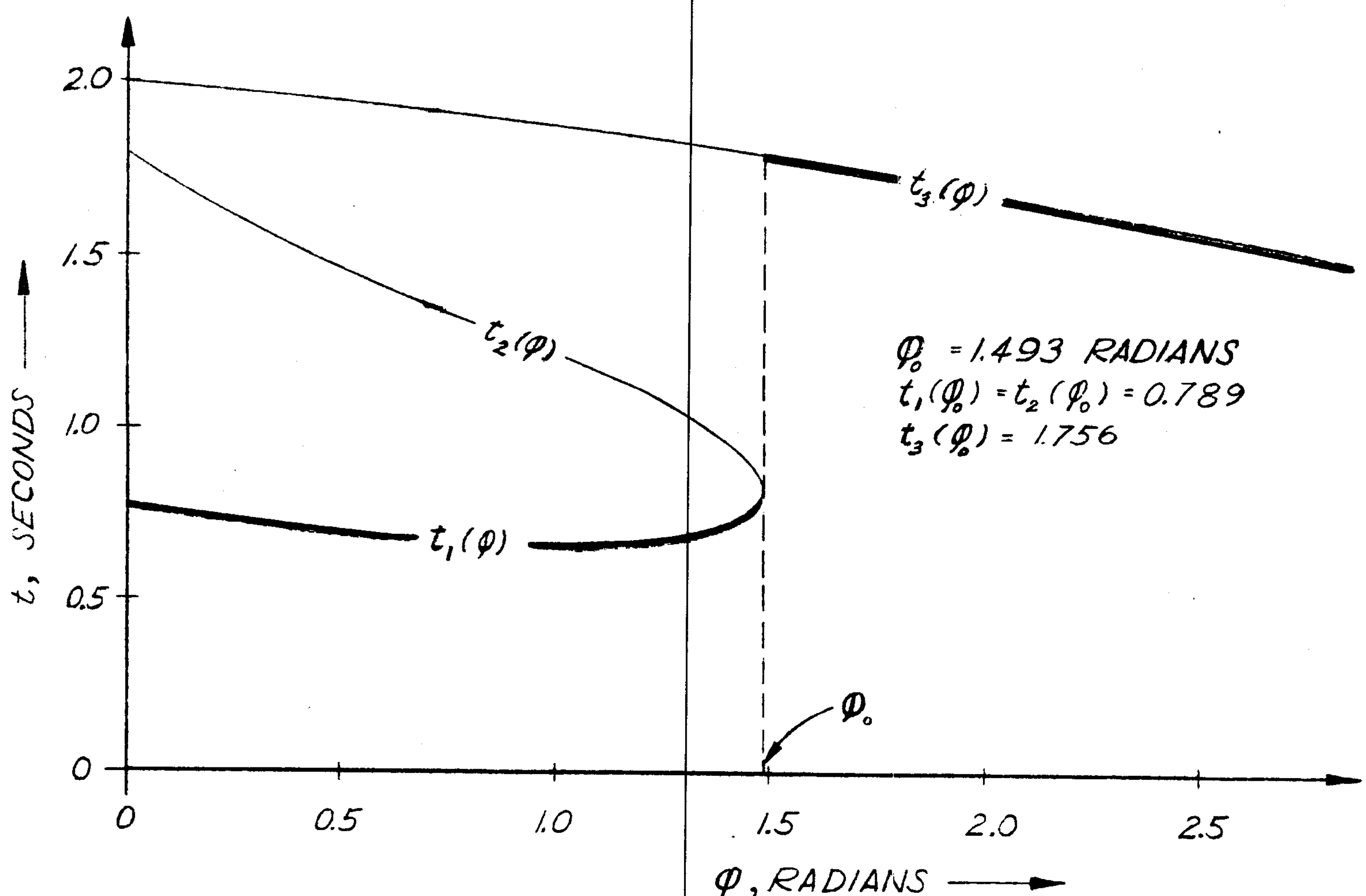

FIG 3.- REAL ROOTS OF $3t^2 - 4t \cos(\pi t + \varphi) - 4 = 0$ IN THE INTERVAL $0 \le t \le 2$